Journal of Statistical Physics, Vol. 53, Nos. 5/6, 1988

Entropic Dimension for Completely Positive Maps

F. Benatti[1] **and H. Narnhofer**[2]

Received June 25, 1988

We extend the concept of quantum dynamical entropy $h_\phi(\gamma)$ to cover the case of a completely positive map γ. For $h_\phi(\gamma)=0$ we examine the limit

$$h_\phi(N, \gamma, \beta) = \lim_n (1/n^\beta)\, H_\phi(N, \gamma N, ..., \gamma^{n-1}N)$$

calling the turning point β_0 between zero and infinite $h_\phi(N, \gamma, \beta)$ the "entropic dimension" $\mathscr{D}_N(\gamma)$. The application of this theory to a solvable irreversible quantum dynamical semigroup on a one-dimensional fermion lattice provides any value of $\mathscr{D}_N(\gamma)$ between 0 and 1.

KEY WORDS: (Quantum) dynamical entropy; completely positive map; contraction semigroup; entropic dimension.

1. INTRODUCTION

Ergodic theory is a well-developed branch in mathematics that originated from the physical motivation to understand thermodynamic behavior, especially the approach to equilibrium. In order to make quantitative statements about the approach to equilibrium, the concept of dynamical entropy turned out to be a useful tool.[1]

However, the approach to equilibrium happens on the microscopic level, which is manifestly non-Abelian, and therefore it was necessary to translate the concept of dynamical entropy also to the quantum case.[2–4] We were able to give a definition and also to calculate the dynamical entropy for the shift and for the free time evolution.[4,5] In ref. 6 the definition was generalized from a one-parameter automorphism group to Z^ν-automorphism groups, making it possible to study space translations together with time translations.

[1] International School for Advanced Studies, Trieste, Italy.
[2] Institut für Theoretische Physik, Universität Wien, Vienna, Austria.

Already for the classical problem, Goldstein[7] argued that the mathematical results are better than the physicist wants them to be, in the sense that the dynamical entropy of the free time evolution is positive. Thus, the free time evolution is perfectly mixing in the sense of the mathematical description, whereas the physicist expects a qualitative improvement of mixing when the system is interacting. Goldstein[7] suggested that in order to observe a qualitative change we have to examine a variation of the dynamical entropy, namely the entropy of the space-time-translation group $\mathbf{R}^4$. But in ref. 6 it is shown that at least for a lattice system we cannot observe anything else but zero dynamical entropy of the whole group $\mathbf{R}^4$. To make a finer distinction between time evolution with vanishing dynamical entropy (so, e.g., no time evolution at all) and time evolution with positive dynamical entropy (with finite group velocity like free time evolution), so as to find a kind of diffusion that spreads like $\sqrt{t}$, we have to change the definition of the dynamical entropy. We consider

$$\lim \frac{1}{n^\beta} H_\phi(\alpha(N),..., \alpha^{n-1}(N))$$

and call this value of β for which the limit changes from 0 to ∞ the entropic dimension of the automorphism with respect to N. For $\beta = 1$ we obtain the usual dynamical entropy.

Of course, we want to check in an example whether such a definition can really be applied. Unfortunately, the only example of time evolutions that are sufficiently explicit to be examined in this context are the quasifree time evolutions, and for these $\beta = 1$. But it is also possible to carry through the concept of dynamical entropy for contraction semigroups and here we will give examples where any value $0 \leqslant \beta \leqslant 1$ can be obtained as entropic dimension.

2. THE DYNAMICAL ENTROPY OF A CONTRACTION MAP

In this section we give the definition of the dynamical entropy of completely positive maps on C^* or von Neumann algebras. To a large extent we only have to copy the results of ref. 4; the generalization from automorphism to c.p. maps are only minor, but for the convenience of the reader we repeat the definitions as far as we shall need them in the following.

We start with a UHF algebra, i.e., the C^*-algebra, that is obtained as the norm limit of an increasing sequence of finite-dimensional matrix subalgebras

$$\mathscr{A} = \overline{\lim_{n\to\infty} \bigcup \mathscr{A}_n}$$

(For generalization to nuclear C^*-algebras see ref. 4.)

Let $N_1,\ldots, N_k$ be finite-dimensional subalgebras of $\mathscr{A}$. Let the state ω be decomposed into states of $\mathscr{A}$

$$\omega = \sum_{i_1,\ldots, i_k} \omega_{i_1,\ldots, i_k} \equiv \sum_I \omega_I, \qquad \omega^l_{i_l} = \sum_{i_1,\ldots, i_k,\, i_l \text{ fixed}} \omega_{i_1,\ldots, i_l,\ldots, i_k} \tag{2.1}$$
$$\hat\omega^l_{i_l} = \omega^l_{i_l}/\omega^l_{i_l}(1)$$

Definition 2.2.

$$H_\omega(N_1,\ldots, N_k) = \sup_{\omega=\sum_I \omega_I} \left[\sum_I -\omega_I(1)\ln\omega_I(1) + \sum_{l=1}^k \sum_{i_l} \omega_{i_l}(1)\ln\omega_{i_l}(1) + \sum_{l=1}^k \sum_{i_l} \omega_{i_l}(1)\, S(\omega\,|\,\hat\omega_{i_l})\,|_{N_l} \right]$$

where $S(\omega\,|\,\hat\omega)\,|_N$ is the relative entropy of the state ω with respect to $\hat\omega$, both restricted to N. Similarly, with γ_l completely positive maps (e.g., the inclusion i_N)

$$H_\omega(\gamma_1,\ldots,\gamma_n) = \sup \sum + \cdots + \sum\sum \omega_{i_l}(1)\, S(\omega\circ\gamma_l\,|\,\hat\omega_{i_l}\circ\gamma_l)$$

Properties 2.3. In ref. 4 the following properties are shown:

1. $0 \leqslant H_\omega(N) \leqslant S(\omega\,|_N)$
2. $H_\omega(N) = H_\omega(N, N)$
3. $H_\omega(N_1,\ldots, N_k) \leqslant H_\omega(\bar N_1,\ldots,\bar N_k)$ for $N_i \subset \bar N_i$
4. $H_\omega(N_1,\ldots, N_k) \leqslant \sum_{i=1}^k H_\omega(N_i)$

We add (compare ref. 2):

5. Let

$$H_\omega(N\backslash\bar N) = \sup_{\omega=\sum\omega_i} \sum_i \omega_i(1)[S(\omega\,|\,\hat\omega_i)\,|_N - S(\omega\,|\,\omega_i)\,|_{\bar N}]$$

Then

$$H_\omega(N_1,\ldots, N_k) \leqslant H_\omega(N_1) + \sum_{i=1}^n H_\omega(N_i\backslash N_{i-1})$$

Proof. Taking for $N_1,\ldots, N_k$ the optimal decomposition, we obtain

$$\begin{aligned} &H_\omega(N_1,\ldots, N_{k-1}, N_k) - H_\omega(N_1,\ldots, N_{k-1}, \bar N_k) \\ &\quad\leqslant \sup_{\omega=\sum\omega_i} \sum_i [S(\omega\,|\,\omega_i)\,|_{N_k} - S(\omega\,|\,\omega_i)\,|_{\bar N_k}] \\ &\quad= \sup_{\omega=\sum\omega_i} \sum_i \omega_i(1)[S(\omega\,|\,\hat\omega_i)\,|_{N_k} - S(\omega\,|\,\hat\omega_i)\,|_{\bar N_k}] \equiv H(N_k\backslash\bar N_k) \end{aligned}$$

Therefore

$$H_\omega(N_1,\ldots, N_k) \leqslant H_\omega(N_1,\ldots, N_{k-1}, N_{k-1}) + H_\omega(N_k \backslash N_{k-1})$$
$$= H_\omega(N_1,\ldots, N_{k-1}) + H_\omega(N_k \backslash N_{k-1})$$

and by induction

$$\leqslant H_\omega(N_1) + \sum_{i=0}^{k-1} H_\omega(N_{i+1} \backslash N_i)$$

6. For $N \subset^\delta \bar{N}$, $\bar{N} \subset^\delta N$, $|H_\omega(N) - H_\omega(\bar{N})| \leqslant c\varepsilon^{1/3}[\ln 2 \cdot d(N)]/\varepsilon$.[4,8] For the definition of $\subset^\delta$, see refs. 2 and 4.

Now we consider a completely positive map γ from $\mathscr{A}$ on $\mathscr{A}$ that satisfies $\omega \circ \gamma = \omega$.

Definition 2.4.

$$h_\omega(N, \gamma) = \lim \frac{1}{n} H_\omega(N,\ldots, \gamma^{n-1} N)$$

Remark. Due to the subadditivity (2.3), (2.4) it suffices to argue that

$$H_\omega(\gamma^n N) = \sup_{\omega = \sum \omega_i} \sum_i \omega_i(1)\, S(\omega \circ \gamma^n \,|\, \hat{\omega}_i \circ \gamma^n)\,|_N$$
$$\leqslant \sup_{\omega = \sum \omega_i} \sum \omega_i(1)\, S(\omega \,|\, \hat{\omega}_i)\,|_N = H_\omega(N)$$

to prove that the limit in (2.4) exists and equals the infimum. The argument runs as follows: We can now use $\omega = \sum_i \tilde{\omega}_i$ as a decomposition for $H_\omega(N)$, where $\tilde{\omega} = \omega_i \circ \gamma^n$. Since the $\tilde{\omega}_i$ do not exhaust all the possible decompositions of ω, $H_\omega(N)$ is bigger than $H_\omega(\gamma^n N)$.

Now we want to get rid of the reference subalgebra N and make the following definition.

Definition 2.5.

$$h_\omega(\gamma) = \sup_{N \text{ finite dimensional}} h_\omega(N, \gamma)$$

But we have even the stronger analogue of the Kolmogoroff–Sinai theorem:

Proposition 2.6. Let N_k be an ascending sequence of finite-dimensional subalgebras, such that

$$\mathscr{A} = \text{n-}\lim_{k \to \infty} \overline{\bigcup N_k} \qquad \text{or} \qquad \mathscr{M} = \pi(\mathscr{A})'' = \text{st-}\lim_{k \to \infty} \overline{\bigcup N_k}$$

Then

$$h_\omega(\gamma) = \lim_k h_\omega(N_k, \gamma) = \sup_k h_\omega(N_k, \gamma)$$

Continuity properties of $H_\omega(N, \gamma)$ are discussed in refs. 4 and 8 in detail. Examining the proof, we have only to use Schwarz positivity (ref. 4, Corollary VI.4) to see if, τ, $\bar{\tau}$ are c.p. maps from $\mathscr{A}$ into $\mathscr{M}$, then

$$\|\gamma^n[\tau(a) - \bar{\tau}(a)]\Omega\| \leqslant \|\tau(a) - \bar{\tau}(a)\Omega\|$$

In ref. 4 it is shown that to any finite-dimensional subalgebra N and ε there exists some N_k such that we can find a c.p. map τ_k from N into N_k with $\|(n - \tau_k n)\Omega\| < \varepsilon$, $\forall n \in N$, and therefore with the continuity estimate of[8]

$$h_\omega(N, \gamma) \leqslant h_\omega(N_k, \gamma) + c\varepsilon^{1/3} \ln \frac{2 \dim N}{\varepsilon}$$

3. THE ENTROPIC DIMENSION OF A C.P. MAP

In refs. 4 and 5 it has been shown that the entropy of space translations coincides with the entropy density. By using equivalence relations, the entropy of any quasifree evolution can be evaluated as strictly positive in one dimension, and infinite in higher dimensions. Evidently, it is against any physical intuition that free time evolution mixes in a qualitatively different way in different dimensions.

On the other hand, we can consider a lattice system in one dimension, $\dim N_x = d$, interacting with a finite-range potential. Then it is known[9] that the system has finite group velocity; therefore

$$\tau_t N_{[0,l]} \overset{\delta}{\subset} N_{[-vt, l+vt]}$$

and, using properties 1, 2, and 6 of the entropic functional,

$$h(\tau, N_{[0,l]}) \leqslant \varepsilon(\delta) + \frac{1}{n} H_\omega(N_{[-vn, l+vn]}) \leqslant 2v \ln d + \varepsilon(\delta)$$

so that the time evolution in one dimension has finite dynamical entropy. On the other hand, imagine a time evolution that behaves like a diffusion process, such that

$$\tau_t N_{[0,l]} \overset{\delta}{\subset} N_{[-v\sqrt{t}, l+v\sqrt{t}]}$$

Then

$$h(\tau, N_{[0,l]}) \leqslant \lim \frac{1}{n} H(N_{[-v\sqrt{n}, l+v\sqrt{n}]}) = 0$$

These observations inspire us to define

Definition 3.1. Let N be a finite-dimensional subalgebra. Let

$$\lim_{n\to\infty} \frac{1}{n^\alpha} H_\omega(N,..., \gamma^n N) = 0, \qquad \alpha > \beta$$

and

$$\lim_{n\to\infty} \frac{1}{n^\alpha} H_\omega(N,..., \gamma^n N) = \infty, \qquad \alpha < \beta$$

Then we call β the entropic dimension of γ with respect to N: $\mathscr{D}_N(\gamma) = \beta$.

Remark. If we consider the space translations of a ν-dimensional lattice, then it is indicated in ref. 6 that

$$\lim \frac{1}{n^\nu} H_\omega(N,..., \sigma^n_{x_1} \cdots \sigma^n_{x_\nu} N) = s(\omega) \neq 0$$

So our definition is in this sense a generalization of the dimension ν. We make the following observations:

Lemma 3.2. The entropic dimension of a one-dimensional semigroup is always $\leqslant 1$.

Lemma 3.3. The entropic dimension of a quasifree evolution τ in any dimension for any finite-dimensional N is 1, so that $\mathscr{D}(\tau) = \sup \mathscr{D}_N(\tau) = 1$.

This follows immediately from the calculations in ref. 5.

Lemma 3.4. The entropic dimension of the identity is zero.

Lemma 3.5. Let γ be a contraction semigroup such that $\omega \circ \gamma = \omega$ and n-lim$[\gamma^n a - \omega(a)] = 0$. Then

$$h_\omega(N, \gamma) = 0 \qquad \text{and} \qquad h_\omega(\gamma) = 0$$

Proof. Due to the norm convergence,

$$\lim_{n\to\infty} \sum \omega_i(1)\, S(\omega \circ \gamma^n \circ i_N \,|\, \hat{\omega}_i \circ \gamma^n \circ i_N) = 0$$

uniformly in all decompositions $\omega = \sum \omega_i$ with $\omega_i(1) > \varepsilon$ (which can be assumed as argued in ref. 4).

Therefore

$$\lim_n \frac{1}{n} H_\omega(N,..., \gamma^n N) \leqslant \lim_n \frac{1}{n} \sum_{k=1}^{n-1} H_\omega(\gamma^k N) = 0 \qquad \text{for any } N$$

4. AN EXAMPLE WITH $\mathscr{D}_N(\gamma) = 0$

Let us be given the Hilbert space

$$l^2(\mathbf{Z}) \equiv \left\{ f \equiv \{f(z)\}_{z \in \mathbf{Z}} : \sum_z |f(z)|^2 < +\infty \right\}$$

Consider the fermion algebra **A** generated on $l^2(\mathbf{Z})$ by the annihilation and creation operators $a^\dagger(f)$, $a(f)$ satisfying

$$\{a^\dagger(f), a(g)\} = \langle f \mid g \rangle = \sum_{z \in \mathbf{Z}} f(z)^* \, g(z)$$

$$\{a(f), a(g)\} = \{a^\dagger(f), a^\dagger(g)\} = 0$$

where f and g belong to $l^2(\mathbf{Z})$, and

$$a(f) = \sum_z f(z) \, a_z, \qquad a_z = a(\delta_z)$$

$$\delta_z(x) = \begin{cases} 1 \cdots z = x \\ 0 \quad \text{otherwise} \end{cases} \qquad \text{or} \qquad \hat{\delta}_z(\theta) = e^{-i\theta_z}$$

Let $\{U_y\}_{y \in \mathbf{Z}}$ be the space-translation group on the one-dimensional lattice:

$$(U_y f)(x) = f_y(x) = f(x - y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, e^{i\theta(x-y)} \hat{f}(\theta)$$

Let Θ be a linear operator on **A** defined as follows:

$$\Theta[X] = \begin{cases} X & \text{if } X \text{ is an even Wick monomial} \\ -X & \text{if } X \text{ is an odd Wick monomial} \end{cases}$$

and $D_S[\cdot]$ be a derivation corresponding to a quasifree time evolution on **A**:

$$D_S\left[\prod_{k=1}^{n} a^{\#}(f_i)\right] = \sum_{k=1}^{n} \prod_{j=1}^{k-1} a^{\#}(f_j) \, a^{\#}(iSf_k) \prod_{j=k+1}^{n} a^{\#}(f_j) \tag{4.1}$$

where

$$\langle p \mid Sf \rangle = \varepsilon(p) \, \hat{f}(p)$$

Let $\{\gamma_t\}_{t \in \mathbf{R}}$ denote the evolution on **A** generated by the following equation of motion:

$$\begin{aligned}\frac{d\gamma_t[X]}{dt} = D_S[\gamma_t[X]] &+ \sum_{z\in\mathbf{Z}} [2a^\dagger(f_z)\,\Theta[\gamma_t[X]]\,a(f_z) \\ &- \{a^\dagger(f_z)\,a(f_z), \gamma_t[X]\}] \\ &+ \sum_{z\in\mathbf{Z}} [2a(g_z)\,\Theta[\gamma_t[X]]\,a^\dagger(g_z) \\ &- \{a(g_z)\,a^\dagger(g_z), \gamma_t[X]\}]\end{aligned} \tag{4.2}$$

$X \in \mathbf{A}$, f and $g \in l^2(\mathbf{Z})$.

Definition 4.3. Given a C^*-algebra $\mathbf{A}$, $\{\gamma_t\}_{t\geqslant 0}$ is called a quantum dynamical semigroup (q.d.s.) if the following conditions are fulfilled:

1. γ_t is a completely positive map on $\mathbf{A}$ for any $t \in \mathbf{R}^+$
2. $\gamma_t \circ \gamma_s = \gamma_{t+s}$ for any t, $s \in \mathbf{R}^+$
3. The semigroup is strongly continuous, i.e.,

$$\|\gamma_t[X] - X\| \xrightarrow{t\to 0^+} 0 \qquad \forall X \in \mathbf{A}$$

4. $\gamma_t(1) = 1$

Theorem 4.4.[10] The rhs of Eq. (4.1) is the generator of a q.d.s. on $\mathbf{A}$ and for $f = g$ the tracial state ϕ is the unique invariant quasifree state under $\{\gamma_t\}_{t\in\mathbf{R}^+}$. Moreover, any other state on $\mathbf{A}$ w^*-converges to ϕ.

Proof. See Theorems 5.1, 5.2, and 8.1 and Segg. in ref. 10.

Remark. Complete positivity is more than positivity, as it concerns tensor products of C^*-algebras, and turns out to be a necessary property of those evolutions of finite systems that originate from a coupling with a reservoir and a subsequent elimination of the unnecessary degrees of freedom.

This procedure has been claimed to be a description of irreversibility in quantum systems and fruitfully applied in several fields of interest.[11]

Lindblad[12] discussed (apart from the fact that he is asking for norm continuity) a more general class of models. Our examples are exactly those for which creation operators are mapped into creation operators, and thus are the generalization of quasifree automorphisms.

Definition 4.5. Let A be the bounded linear operator on $l^2(\mathbf{Z})$ defined by

$$A = \sum_{z\in\mathbf{Z}} |f_z\rangle\langle f_z|, \qquad \langle\theta|\,A\,|h\rangle = |\hat{f}(\theta)|^2\,\hat{h}(\theta) \tag{4.6}$$

Set

$$\gamma(\theta) \equiv |\hat{f}(\theta)|^2 \qquad \text{and} \qquad C = iS - 2A: \tag{4.7}$$

$$\langle \theta | C | h \rangle = [i\varepsilon(\theta) - 2\gamma(\theta)]\, \hat{h}(\theta) \tag{4.8}$$

Denote $a^{\#}(e^{Ct}\delta_r)$ with $a_r^{\#}(t)$, where

$$(e^{Ct}\delta_r)(\theta) = e^{i\varepsilon(\theta)t - 2\gamma(\theta)t} e^{-ir\theta}$$

Let $\{i\}_k^p$ ($\overline{\{i\}}_k^p$) be a subdivision of p integers into two subgroups of $p-k$ (k) and k ($p-k$) elements, both arranged in increasing order.

Let $\varepsilon(\{i\}_k^p)$ [$\varepsilon(\overline{\{i\}}_k^p)$] be the sign of $\{i\}_k^p$ ($\overline{\{i\}}_k^p$) as a permutation of the initial p integers.

Call

$$F^t_{\{i\}_k^p, \overline{\{j\}}_k^q}$$

the determinant of the matrix

$$\begin{bmatrix} F^t_{i_k j_1} & \cdots & F^t_{i_k j_k} \\ \vdots & & \vdots \\ F^t_{i_1 j_1} & \cdots & F^t_{i_1 j_k} \end{bmatrix}$$

where

$$F^t_{rs} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left[\frac{1 - e^{4\gamma(\theta)t}}{2} \right] e^{i\theta(r-s)}$$

Theorem 4.9. Given the ordered Wick monomial

$$X = \prod_{r=1}^{p} a^{\dagger}_{i_r} \prod_{s=1}^{q} a_{j_s}$$

where i_r and j_s denote points on the lattice, we have

$$\gamma_t[X] = \sum_{k=0}^{[q,p]} \sum_{\{n\}_k^p} \sum_{\overline{\{m\}}_k^q} \varepsilon(\{n\}_k^p)\, \varepsilon(\overline{\{m\}}_k^q)\, F^t_{\{n\}_k^p, \overline{\{m\}}_k^q} \prod_{r=k+1}^{p} a^{\dagger}_{n_r}(t) \prod_{s=k+1}^{q} a_{m_s}(t)$$

where

$$[q, p] \equiv \min\{q, p\}$$

$$\{n\}_k^p \equiv \{n_{k+1} < \cdots < n_p,\, n_1 < \cdots < n_k,\, n_r \in (i_1, \ldots, i_p)\}$$

$$\overline{\{m\}}_k^q \equiv \{m_1 < \cdots < m_k,\, m_{k+1} < \cdots < m_q,\, m_s \in (j_1, \ldots, j_q)\}$$

Proof. See Appendix A.

Definition 4.10. By duality, $\{\gamma_t\}_{t\geqslant 0}$ induces an evolution $\{\gamma_t^*\}_{t\geqslant 0}$ on the state space: if ω is a state on **A**, then

$$\omega_t[X] = \gamma_t^*\omega[X] = \omega[\gamma_t[X]]$$

The invariant state ϕ, if any, will be called uniformly stable if for any other state ω we have that

$$\lim_{t\to+\infty} \|\omega_t - \phi\| = 0$$

Remark. The algebra **A** we are dealing with is a UHF algebra. It is given by the norm closure of an ascending sequence of matrix subalgebras M_d of increasing dimension 2^d, i.e., those matrix algebras generated by annihilation and creation operators sitting at the points of a d-point interval around the origin.

Theorem 4.11. Let a M_d subalgebra in the generating sequence $\{M_d\}$ be fixed. If $\gamma(\theta)$ is strictly positive (e.g., a nonzero constant), then the state obtained by restricting the tracial state ϕ to M_d is uniformly stable with respect to the restriction of any other state ω on **A** to the same M_d. This holds true for any $d< +\infty$ together with

$$\|\gamma_t^*\omega\,|_{M_d} - \phi\,|_{M_d}\| \leqslant e^{-2\gamma t} f(d)$$

if

$$\gamma \equiv \min \gamma(\theta), \qquad \theta\in[-\pi,\pi]$$

Proof. See Appendix B.

Remark. $f(d) = 6^d d!$ provides for an estimate which is surely non-optimal and which cannot be extended to cover the case of finite-dimensional subalgebras not contained in any of the M_d. Theorem 4.11 improves in this simple model the weak* convergence shown by Davies[10] insofar as it gives an explicit bound when w^*-convergence and norm convergence coincide.

Corollary 4.12. $h_\phi(\gamma, M_d, \beta) = 0$ for any $d< +\infty$ and any $0<\beta\leqslant 1$.

Proof. Since $\{\gamma_t\}_{t\geqslant 0}$ is a q.d.s., we can apply the results reported in the preceding section. Using the subadditivity of the H_ϕ functional, where ϕ is now the tracial state, we get

$$H_\phi(M_d, \gamma M_d,..., \gamma^{n-1} M_d) \leqslant \sum_{k=0}^{n-1} H_\phi(\gamma^k M_d)$$

Using the uniform estimate of Theorem 4.2, the inequality[8]

$$|S_\omega - S_{\bar{\omega}}| \leqslant \varepsilon \ln \frac{2d}{\varepsilon}$$

when ω and $\bar{\omega}$ are states on a d-dimensional algebra such that $\|\omega - \bar{\omega}\| = \varepsilon$, and the definition of $H_\phi(\gamma^k M_d)$, we finally obtain

$$\lim_n \frac{1}{n^\beta} H_\phi(M_d, \ldots, \gamma^{n-1} M_d) \leqslant \lim_n \frac{1}{n^\beta} \sum_{k=N}^{n-1} f(d)\, e^{-2\gamma k} \left[\ln \frac{2^{2d+1}}{f(d)} + 2\gamma k \right]$$

for $N \gg 1$. This limit is zero for any $0 < \beta$.

Corollary 4.13.

$$h_\phi(\gamma) = \sup_{M \text{ finite dimensional}} h_\phi(M, \gamma, \beta = 1) = 0$$

Proof. We can apply the non-Abelian analogue of the Kolmogoroff–Sinai theorem and the result follows simply by Corollary 4.12.

Remark. In order to say something about $h_\phi(\gamma, \beta)$, we should be able to manage $h_\phi(N, \gamma, \beta)$ when N is an arbitrary finite-dimensional subalgebra. In other words, we should need a control on the stability of the tracial state ϕ with respect to the restriction on any $N \subset \mathbf{A}$ and not only to the matrix subalgebras M_d. This is not necessary for the $\beta = 1$, since then norm-equicontinuity in the $\gamma^k \circ i_N$ maps appearing in the H_ϕ functional allows us to avoid seeking a uniform bound for the difference $\|\gamma_t(x) - \phi(x)\|$ in Theorem 4.11.

5. ENTROPIC DIMENSION $0 < \beta < 1$

In this section we vary the example of Section 4 by allowing that $\gamma(\theta) = 0$, somewhere between $-\pi$ and π, and we take $\gamma(0) = 0$. We expect that the entropic dimension is given by the way $\gamma(\theta)$ tends to zero for $\theta \to 0$ compared to the mixing behavior of the norm-preserving part of the contraction map that is $\varepsilon(\theta)$ in the corresponding neighborhood of $\theta = 0$. For simplicity we fix our initial algebra $N = \mathcal{M}_1 \equiv$ the 2×2 matrix algebra generated by a_0 and $a_0^\dagger$, the annihilation and creation operators at the origin.

For this matrix we denote:

Definition 5.1. τ is the tracial state on $\mathcal{M}_1$ corresponding to the density matrix $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For an arbitrary state ϕ_i, $\hat{\phi}_i \circ \gamma_t \circ i_0$ corresponds to a density matrix

$$\rho_{i,t} = \begin{pmatrix} a_{i,t} & b_{i,t} \\ \overline{b_{i,t}} & 1 - a_{i,t} \end{pmatrix}$$

where $a_{i,t}=\hat{\phi}_i(\gamma_t(a_0^\dagger a_0))$, $b_{i,t}=\hat{\phi}_i(\gamma_t(a_0))$, and $i_0 \equiv i_{\mathcal{M}_1}$. We obtain the following results.

Theorem 5.2. Assume that γ is the contraction semigroup defined in (4.3)–(4.5). Assume that $i\varepsilon(\theta)=i\theta$ (i.e., the shift). Assume that $\gamma(\theta)$ approaches 0 as $|\theta|^\alpha$. Then

$$\mathscr{D}_{\mathcal{M}_1}(\gamma)=(\alpha-2)/\alpha, \qquad \alpha>2$$

Proof. We are interested in the asymptotic dependence on n of $H_\phi(\mathcal{M}_1,..., \gamma^{n-1}\mathcal{M}_1)$ and will control it by finding lower and upper bounds with the same leading term $n^{\alpha/(\alpha-2)}$ for large n.

Lower Bound. From the definition of H_ϕ we get a lower bound just by fixing a decomposition of the invariant state ϕ.

We choose

$$x_{(i)}=x_{i_0\cdots i_{n-1}}=\prod_{k=0}^{n-1}\frac{1+(-1)^{i_k}(a_k^\dagger+a_k)}{2}, \qquad i_k=0,1$$

and

$$\hat{\phi}_{(i)}=\phi(x_{(i)})^{-1}\,\phi(x_{(i)}\cdot)$$

Then, due to the orthogonality of $\delta_k(\theta)=e^{-i\theta k}$, the tracial state factorizes such that

$$\phi(x_{(i)})=\prod_{k=0}^{n-1}\phi\left(\frac{1+(-1)^{i_k}(a_k+a_k^\dagger)}{2}\right)=\frac{1}{2^n}$$

and

$$\phi_{i_k}^k(\cdot)=\phi\left(\frac{1+(-1)^{i_k}(a_k+a_k^\dagger)}{2}\cdot\right) \tag{5.3}$$

The classical part of the entropic functional H_ϕ vanishes due to the factorization, and we remain with

$$\begin{aligned}H_\phi(\mathcal{M}_1,..., \gamma^{n-1}\mathcal{M}_1) &\geqslant \sum_{k=0}^{n-1}\phi_{i_k}^k(1)\,S(\bar{\phi}_{\mathcal{M}_1}\,|\hat{\phi}_{i_k}^k\circ\gamma^k|_{\mathcal{M}_1})\\ &=\frac{1}{2}\sum_{k=0}^{n-1}\sum_{i_k}S(\tau\,|\,\rho_{i_k,k})=nS(\tau)-\frac{1}{2}\sum_{k=0}^{n-1}\sum_{i_k}S(\rho_{i_k,k})\end{aligned}$$

The entropy $S(\rho)$ is determined by the eigenvalues of ρ, which we know sufficiently explicitly:

$$a_{i_k,k} = \frac{1}{2}$$

$$b_{i_k,k} = (-1)^{i_k} \frac{1}{4\pi} \int_{-\pi}^{+\pi} d\theta \, e^{-2\gamma(\theta)k} \equiv \frac{(-1)^{i_k}}{2} x_k \tag{5.4}$$

$$\rho^{1,2}_{i_k,k} = \frac{1}{2} \pm \frac{x_k}{2}$$

$$\begin{aligned} \frac{1}{2} \sum_{i_k} S(\rho_{i_k,k}) - \ln 2 &= -\frac{1+x_k}{2} \ln \frac{1+x_k}{2} - \frac{1-x_k}{2} \ln \frac{1-x_k}{2} - \ln 2 \\ &= O(x_k^2) = O(k^{-2/\alpha}) \end{aligned} \tag{5.5}$$

due to the following result.

Proposition 5.6. Let us assume that $\gamma(\theta) \overset{\theta \to 0}{\sim} |\theta|^\alpha$. Then,

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, e^{-2\gamma(\theta)k} \overset{k \to +\infty}{\sim} \frac{1}{k^{1/\alpha}}$$

Proof. We choose $\delta > 0$ so small that

$$x_k = \frac{1}{2\pi} \int_{-\delta}^{+\delta} d\theta \, e^{-2|\theta|^\alpha k} + O(e^{-c_\delta k})$$

(c_δ is a strictly positive constant depending on $\delta > 0$).

Since $\sum_{k=1}^n k^{-2/\alpha} \geqslant cn^{(\alpha-2)/\alpha}$ with $c > 0$ for n large enough, we have

$$\lim n^{\alpha/(\alpha-2)} [H_\phi(\mathcal{M}_1, ..., \gamma^{n-1}\mathcal{M}_1) - \underline{c} n^{(\alpha-2)/\alpha}] > 0 \qquad \text{for some} \quad \underline{c} > 0$$

Upper Bound. We use the upper bound for $H_\phi(\mathcal{M}_1, ..., \gamma^{n-1}\mathcal{M}_1)$ [(2.1), (2.5)] and are therefore interested in the k dependence of $H_\phi(\gamma^k\mathcal{M}_1 \backslash \gamma^{k-1}\mathcal{M}_1)$.

Proposition 5.7. Assume that $\lim \gamma^k m = \phi(m)$ for $m \in \mathcal{M}_1$. Then the leading term of $H_\phi(\gamma^k\mathcal{M}_1 \backslash \gamma^{k-1}\mathcal{M}_1)$ for large k is given by

$$\sup_{\phi = \sum \phi_i} \sum \phi_i(1)(\|\rho_{i,k} - \tau\|_2^2 - \|\rho_{i,k-1} - \tau\|_2^2)$$

where $\|\cdot\|_2$ is the Hilbert–Schmidt norm.

Proof. See Appendix C.

The eigenvalues of the Hermitian matrix $\rho_{i,k} - \tau$ are

$$\lambda_\pm = \pm(|b_{i,k}|^2 + |\tfrac{1}{2} - a_{i,k}|^2)^{1/2}$$

We denote $\delta_0^k = e^{Ck}\delta_0$ and $a(\delta_0^k) = a_0(k)$,

$$\begin{aligned}
&\|\rho_{i,k} - \tau\|_2^2 - \|\rho_{i,k-1} - \tau\|_2^2 \\
&\quad = 2(|b_{i,k}|^2 - |b_{i,k-1}|^2 + a_{i,k}^2 - a_{i,k-1}^2 - a_{i,k-1} + a_{i,k+1}) \\
&\quad = 2\Bigg\{|\hat{\phi}_i(a_0(k))|^2 - |\hat{\phi}_i(a_0(k-1))|^2 + \hat{\phi}_i(a_0^\dagger(k)\, a_0(k))^2 \\
&\qquad - \hat{\phi}_i(a_0^\dagger(k-1)\, a_0(k-1))^2 + \frac{1}{4}(1 - \|\delta_0^k\|^2)^2 - \frac{1}{4}(1 - \|\delta_0^{k-1}\|^2)^2 \\
&\qquad + \hat{\phi}_i(a_0^\dagger(k)\, a_0(k))(1 - \|\delta_0^k\|^2) - \hat{\phi}_i(a_0^\dagger(k-1)\, a_0(k-1))(1 - \|\delta_0^{k-1}\|^2) \\
&\qquad + \hat{\phi}_i(a_0^\dagger(k-1)\, a_0(k-1)) - \hat{\phi}_i(a_0^\dagger(k)\, a_0(k)) + \frac{\|\delta_0^k\|^2}{2} - \frac{\|\delta_0^{k-1}\|^2}{2}\Bigg\}
\end{aligned}$$

Therefore (5.8)

$$\begin{aligned}
&\sum \phi_i(1)(\|\rho_{i,k} - \tau\|_2^2 - \|\rho_{i,k-1} - \tau\|_2^2) \\
&\quad = 2\sum \phi_i(1)\Bigg[\,|\hat{\phi}_i(a_0(k))|^2 - |\hat{\phi}_i(a_0(k-1))|^2 && \text{(1st term)} \\
&\qquad + \hat{\phi}_i(a_0^\dagger(k)\, a_0(k))^2 - \hat{\phi}_i(a_0^\dagger(k-1)\, a_0(k-1))^2 && \text{(2nd term)} \\
&\qquad + \frac{\|\delta_0^{k-1}\|^4 - \|\delta_0^k\|^4}{4}\Bigg] && \text{(3rd term)}
\end{aligned}$$

We are now interested in studying the behavior for large k of the above three terms.

In order to get an upper bound, they should be given independently of any special decomposition of ϕ.

1st Term.

$$\begin{aligned}
&|\hat{\phi}_i(a_0(k))|^2 - |\hat{\phi}_i(a_0(k-1))|^2 \\
&\quad = \hat{\phi}_i(a_0(k))\, \hat{\phi}_i(a_0^\dagger(k) - a_0^\dagger(k-1)) + \hat{\phi}_i(a_0^\dagger(k-1))\, \hat{\phi}_i(a_0(k) - a_0(k-1)) \\
&\quad \leqslant \|\delta_0^k - \delta_0^{k-1}\|\, (\|\delta_0^k\| + \|\delta_0^{k-1}\|) = O\left(\frac{1}{k^{3/2\alpha}}\right) O(k^{-1/2\alpha}) = O(k^{-2/\alpha})
\end{aligned}$$

For the last estimate see Proposition 5.9 below.

2nd Term.

$$
\begin{aligned}
&\hat{\phi}_i(a_0^\dagger(k)\, a_0(k))^2 - \hat{\phi}_i(a_0^\dagger(k-1)\, a_0(k-1))^2 \\
&\quad = [\hat{\phi}_i(a_0^\dagger(k)\, a_0(k)) + \hat{\phi}_i(a_0^\dagger(k-1)\, a_0(k-1))] \\
&\qquad \times \{\hat{\phi}_i(a_0^\dagger(k)[a_0(k) - a_0(k-1)]) + \hat{\phi}_i([a_0^\dagger(k) - a_0^\dagger(k-1)]\, a_0(k-1))\} \\
&\quad \leqslant (\|\delta_0^k\|^2 + \|\delta_0^{k-1}\|^2)(\|\delta_0^k\| + \|\delta_0^{k-1}\|)\, \|\delta_0^k - \delta_0^{k-1}\| \\
&\quad = O(k^{-1/\alpha})\, O(k^{-1/2\alpha})\, O(k^{-3/2\alpha}) \\
&\quad = O(k^{-3/\alpha})
\end{aligned}
$$

3rd Term.

$$
\begin{aligned}
&\tfrac{1}{4}(\|\delta_0^{k-1}\|^2 - \|\delta_0^k\|^2)(\|\delta_0^{k-1}\|^2 + \|\delta_0^k\|^2) \\
&\qquad = O(k^{-(1+\alpha)/\alpha})\, O(k^{-1/\alpha}) = O(k^{-(2+\alpha)/\alpha})
\end{aligned}
$$

From Proposition 5.7 we already know that

$$\|\delta_0^k\| \overset{k \to +\infty}{\sim} \frac{1}{k^{1/2\alpha}}$$

Therefore $\|\delta_0^k - \delta_0^{k-1}\|$ and $\|\delta_0^{k-1}\| - \|\delta_0^k\|$ need to be estimated:

Proposition 5.9. Set $\alpha > 1$; then

$$\|\delta_0^{k-1}\|^2 - \|\delta_0^k\|^2 \overset{k \to +\infty}{\sim} (1/k)^{(1+\alpha)/\alpha}$$

and

$$\|\delta_0^k - \delta_0^{k-1}\| \overset{k \to +\infty}{\sim} (1/k)^{3/2\alpha}$$

Proof. As was previously done, we can consider a neighborhood of $\theta = 0$ so small that $\gamma(\theta)$ can be identified with $|\theta|^\alpha$. Outside this region the integral decays exponentially fast with k.

Hence $\|\delta_0^{k-1}\|^2 - \|\delta_0^k\|^2$, for large k, goes like

$$\int_{-\delta}^{\delta} d\theta\, e^{-4|\theta|^\alpha k}[1 + 4|\theta|^\alpha - 1] \sim (1/k)^{(1+\alpha)/\alpha}$$

and $\|\delta_0^k - \delta_0^{k-1}\|^2$ goes like

$$
\begin{aligned}
&\int_{-\delta}^{\delta} d\theta\, e^{-4|\theta|^\alpha k}(1 + e^{4|\theta|^\alpha} - 2e^{2|\theta|^\alpha}\cos\theta) \\
&\qquad \sim \int_{-\delta}^{\delta} d\theta\, e^{-4|\theta|^\alpha k}\theta^2 \sim (1/k)^{3/\alpha} \qquad \text{if} \quad \alpha > 1
\end{aligned}
$$

Conclusions. For any decomposition of ϕ into the states $\hat{\phi}_i$ the first term goes to zero faster than $(1/k)^{2/\alpha}$, the second one goes to zero faster than $(1/k)^{3/\alpha}$, and the third term goes to zero faster than $(1/k)^{(2+\alpha)/\alpha}$.

Evidently the first term is the leading one so that $\exists$ some c such that

$$\lim k^{\alpha/2}[H(\gamma^k \mathcal{M}_1 \backslash \gamma^{k-1}\mathcal{M}_1) - ck^{-\alpha/2}] < 0$$

and

$$\lim n^{\alpha/(2-\alpha)} H_\phi(\mathcal{M}_1, \ldots, \gamma^{n-1}\mathcal{M}_1) \leqslant \bar{c}$$

This proves Theorem 5.1.

Remark. Since we already know the lower bound for H_ϕ, we notice that there must exist a decomposition $\{\phi_i\}$ such that

$$|\hat{\phi}_i(a_0(k))|^2 - |\hat{\phi}_i(a_0(k-1))|^2 \sim (1/k)^{2/\alpha}$$

We try a decomposition

$$\hat{\phi}_i(\cdot) = \phi\left(\frac{1+b^\dagger+b}{2}\cdot\right)$$

and take for the normalized function corresponding to b

$$f(\theta) = \frac{e^{i\theta k - 2\gamma(\theta)k} g(\theta)}{\|\cdot\|}$$

We notice first that $g(\theta) = 1$ gives the optimal decomposition if we want to calculate $H_\phi(\gamma_k \circ i_0)$: Using the result of (5.2), we have to evaluate

$$O(x_k) = O(\langle f \mid \delta_0^k \rangle / \| f \|)$$

so the optimal choice is

$$f = \delta_0^k / \|\delta_0^k\|$$

For evaluating $H_\phi(\gamma^k \circ i_0 \backslash \gamma^{k-1} \circ i_0)$ the leading term is given by

$$\begin{aligned}
&\langle f \mid \delta_0^k \rangle \langle \delta_0^k \mid f \rangle - \langle f \mid \delta_0^{k-1} \rangle \langle \delta_0^{k-1} \mid f \rangle \\
&= \langle f \mid \delta_0^k - \delta_0^{k-1} \rangle \langle \delta_0^k \mid f \rangle + \langle f \mid \delta_0^{k-1} \rangle \langle \delta_0^k - \delta_0^{k-1} \mid f \rangle \\
&\simeq \int d\theta_1 \, [\exp(-2k\theta_1^\alpha)] \, \bar{g}(\theta_1) \left(i\theta_1 + \frac{\theta_1^2}{2} \cdots\right) \int [\exp(-2k\theta_2^\alpha)] \, \bar{\bar{g}}(\theta_2) \, d\theta_2 \\
&\quad + \int \bar{g}(\theta_2)[\exp(-2k\theta_2^\alpha)] \, d\theta_2 \int \left(-i\theta_1 + \frac{\theta_1^2}{2} - \theta_1\theta_2 \cdots\right) [\exp(-2k\theta_1^\alpha)] \bar{\bar{g}}(\theta_1) \, d\theta_1
\end{aligned}$$

The integration with the $i\theta$ term is dominating, but only remains for $\tilde{g}(\theta) \neq 1$ in the appropriate scaling, which shows how delicate the decomposition for $H_\phi(\gamma^k \circ i_0 \backslash \gamma^{k-1} \circ i_0)$ has to be.

We return to our initial problem and state the following result.

Theorem 5.10. Assume that γ is again the contraction semigroup of (4.3)–(4.5). Assume that $\gamma(\theta)$ approaches 0 as $|\theta|^\alpha$ and $\varepsilon(\theta)$ approaches 0 as $|\theta|^\beta$. Then

$$\mathscr{D}_{\mathscr{M}_1}(\gamma) = \frac{\alpha - 1 - \beta}{\alpha}$$

The proof can be found in ref. 13. We have only to vary the ideas to prove Theorem 5.1: For the lower bound we start with a function so that all $e^{iS_n}f$ are orthogonal to each other and thus we control the classical part. The upper bound remains the same; only the integrals have to be estimated differently.

To discuss the above result, we observe the following: Our contraction semigroup can be split into two parts, the pure contraction $\gamma(\theta)$ and the automorphism $\varepsilon(\theta)$; both let the state converge, $\gamma(\theta)$ strongly and $\varepsilon(\theta)$ weakly. Nevertheless, in the dynamical entropy they are in competition; $\varepsilon(\theta)$ increases the entropy, the better it mixes, while $\gamma(\theta)$ decreases the entropy, the faster it lets the state converge. This shows, mentioned in the introduction, that the relation between the mathematical concept of dynamical entropy and the physical problem, how a system approaches equilibrium, needs further investigation.

APPENDIX A

Given the fermionic algebra **A** on the Hilbert space $l^2(\mathbf{Z})$, let us consider:

(a) An ordered Wick monomial

$$X \equiv \prod_{i=1}^{p} a^\dagger(f_i) \prod_{j=1}^{q} a(g_j)$$

(f_i, g_j belong to $l^2(\mathbf{Z})$ $\forall i, j$).

(b) The bounded operators on $l^2(\mathbf{Z})$

$$A = \sum_{z \in \mathbf{Z}} |F_z\rangle\langle F_z|, \qquad B = \sum_{z \in \mathbf{Z}} |G_z\rangle\langle G_z|$$

[F, G belong to $l^2(\mathbf{Z})$ and $F_z(x) = F(x-z)$, $G_z(x) = G(x-z)$].

Define two bounded linear operators $W_A[\cdot]$ and $Y_B[\cdot]$ on $\mathbf{A}$ as follows:

$$W_A\left[\prod_{i=1}^{p} a^\dagger(f_i) \prod_{j=1}^{q} a(g_j)\right] = -\sum_{k=1}^{p} \prod_{i=1}^{k-1} a^\dagger(f_i)\, a^\dagger(Af_k) \prod_{i=k+1}^{p} a^\dagger(f_i) \prod_{j=1}^{q} a(g_j) - \sum_{k=1}^{q} \prod_{i=1}^{p} a^\dagger(f_i) \prod_{j=1}^{k-1} a(g_j)\, a(Ag_k) \prod_{j=k+1}^{q} a(g_j) \tag{A.1}$$

$$Y_B\left[\prod_{j=1}^{q} a(g_j) \prod_{i=1}^{p} a^\dagger(f_i)\right] = -\sum_{k=1}^{q} \prod_{j=1}^{k-1} a(g_j)\, a(Bg_k) \prod_{j=k+1}^{q} a(g_j) \prod_{i=1}^{p} a^\dagger(f_i) - \sum_{k=1}^{p} \prod_{j=1}^{q} a(g_j) \prod_{i=1}^{k-1} a^\dagger(f_i)\, a^\dagger(Bf_k) \prod_{i=k+1}^{p} a^\dagger(f_i) \tag{A.2}$$

Let $D_S[\cdot]$ be the generator of the free evolution on $\mathbf{A}$, i.e.,

$$D_S\left[\prod_{i=1}^{p} a^\dagger(f_i) \prod_{j=1}^{q} a(g_j)\right] = \sum_{k=1}^{p} \prod_{i=1}^{k-1} a^\dagger(f_i)\, a^\dagger(iSf_k) \prod_{i=k+1}^{p} a^\dagger(f_i) \prod_{j=1}^{q} a(g_j) + \sum_{k=1}^{q} \prod_{i=1}^{p} a^\dagger(f_i) \prod_{j=1}^{k-1} a(g_j)\, a(iSg_k) \prod_{j=k+1}^{q} a(g_j) \tag{A.3}$$

In ref. 10 it is shown that

$$L[X] = D_S[X] + W_A[X] + Y_B[X]$$

is the generator of a quantum dynamical semigroup $\{\gamma_t\}_{t\geqslant 0}$ on $\mathbf{A}$ which actually coincides with that introduced in Section 4. Set

$$C = iS - A - B, \qquad B_0 = B + B^\dagger$$

$$F_{ij}^t = \int_0^t ds(e^{Cs}f_i, B_0 e^{Cs} g_j), \qquad [q, p] = \min\{q, p\}$$

Given p (q) integers in increasing order, $\{i\}_k^p$ $(\overline{\{j\}}_k^q)$ indicates the permutation obtained by grouping together $p-k$ (k) and k ($q-k$) elements, respectively, and arranging them in increasing order:

$$\{i\}_k^p \equiv (i_{k+1},..., i_p, i_1,..., i_k): \quad i_{k+1} < \cdots < i_p, i_1 < \cdots < i_k$$

$$\overline{\{j\}}_k^q \equiv (j_1,..., j_k, j_{k+1},..., j_q): \quad j_1 < \cdots < j_k, j_{k+1} < \cdots < j_q$$

$\varepsilon(\{i\}_k^p)$ and $\varepsilon(\overline{\{j\}}_k^q)$ are the signs of the corresponding permutations. Let

$$F^t_{\{i\}_k^p, \overline{\{j\}}_k^q}$$

be the determinant of

$$\begin{bmatrix} F^t_{i_k j_1} & \cdots & F^t_{i_k j_k} \\ \vdots & & \vdots \\ F^t_{i_1 j_1} & \cdots & F^t_{i_1 j_k} \end{bmatrix}$$

and set

$$F^t_{\{i\}_0^p, \overline{\{j\}}_0^q} \equiv 1$$

The proof of the statement of Theorem 4.1 consists of two steps. First we show that

$$\begin{aligned} L[X] = &\sum_{k=1}^{p} \prod_{i=1}^{k-1} a^\dagger(f_i)\, a^\dagger(Cf_k) \prod_{i=k+1}^{p} a^\dagger(f_i) \prod_{j=1}^{q} a(g_j) \\ &+ \sum_{k=1}^{q} \prod_{i=1}^{p} a^\dagger(f_i) \prod_{j=1}^{k-1} a(g_j)\, a(Cg_k) \prod_{j=k+1}^{q} a(g_j) \\ &+ \sum_{\{i\}_1^p \overline{\{j\}}_1^q} \varepsilon(\{i\}_1^p)\, \varepsilon(\overline{\{j\}}_1^q)(f_{i_1}, B_0\, g_{j_1}) \cdot \prod_{r=2}^{p} a^\dagger(f_{i_r}) \prod_{s=2}^{q} a(g_{j_s}) \end{aligned} \tag{A.4}$$

Then we show that

$$\begin{aligned} \gamma_t[X] = &\sum_{k=0}^{[q,p]} \sum_{\{i\}_k^p} \sum_{\overline{\{j\}}_k^q} \varepsilon(\{i\}_k^p)\, \varepsilon(\overline{\{j\}}_k^q)\, F^t_{\{i\}_k^p, \overline{\{j\}}_k^q} \\ &\times \prod_{r=k+1}^{p} a^\dagger(e^{Ct} f_{i_r}) \prod_{s=k+1}^{q} a(e^{Ct} g_{j_s}) \end{aligned} \tag{A.5}$$

satisfies the evolution equation

$$\frac{d}{dt} \gamma_t[X] \equiv L[\gamma_t[X]]$$

Although not very complicated, the proof is rather long and tedious. We skip most of the details and sketch the main lines.

Taking (A.4) for granted, we find that (A.5) is true just by comparing

$$\sum_{k=0}^{[q,p]} \sum_{\{i\}_k^p} \sum_{\overline{\{j\}_k^q}} \varepsilon(\{i\}_k^p)\, \varepsilon(\overline{\{j\}_k^q}) \frac{d}{dt} F^t_{\{i\}_k^p, \overline{\{j\}_k^q}} \prod_{r=k+1}^{p} a^\dagger(e^{Ct} f_{i_r}) \prod_{s=k+1}^{q} a(e^{Ct} g_{j_s}) \tag{A.6}$$

which comes out from $d\gamma_t[X]/dt$, with

$$\sum_{k=0}^{[q,p]} \sum_{\{i\}_k^p} \sum_{\overline{\{j\}_k^q}} \varepsilon(\{i\}_k^p)\, \varepsilon(\overline{\{j\}_k^q})\, F^t_{\{i\}_k^p, \overline{\{j\}_k^q}}$$
$$\times \left\{ \sum_{n=k+1}^{p} \sum_{m=k+1}^{q} \varepsilon(i_{k+1},\dots, i_p, i_n)\, \varepsilon(j_m, j_{k+1},\dots, j_q) \right.$$
$$\left. \times (e^{Ct} f_{i_n}, B_0 E^{Ct} g_{i_m}) \prod_{r=k+1}^{p}{}' a^\dagger(e^{Ct} f_{i_r}) \prod_{s=k+1}^{q}{}' a(e^{Ct} g_{j_s}) \right\} \tag{A.7}$$

which we get by applying $L[\cdot]$ to $\gamma_t[X]$. [The primes mean that $a^\dagger(e^{Ct} f_{i_k})$ and $a(e^{Ct} g_{j_m})$ are to be removed from $\prod_{r=k+1}^{p} a^\dagger(e^{Ct} f_{i_r})$ and $\prod_{s=k+1}^{q} a(e^{Ct} g_{j_s})$, respectively.]

Observe that the sum in (A.6) starts from $k=1$ due to the definition of

$$F^t_{\{i\}_k^p, \overline{\{j\}_k^q}}$$

and that the sum in (A.7) ends with $k=[q,p]-1$ due to (A.4).

After setting $k=q+1$ in (A.6) and fixing in both formulas the coefficients of equal products

$$\prod_{a=k+2}^{p} a^\dagger(e^{Ct} f_{r_a}) \prod_{b=k+2}^{q} a(e^{Ct} g_{s_b})$$

(A.5) is a consequence of

$$(e^{Ct} f_{i_n}, B_0 e^{Ct} g_{j_m}) = \frac{d}{dt} F^t_{i_n j_m}$$

and

$$\frac{d}{dt} F^t_{\{r\}_{k+1}^p, \overline{\{s\}_{k+1}^q}} = \sum_{n=1}^{k+1} \sum_{m=1}^{k+1} \varepsilon(r_n, r_1,\dots, r_{k+1})\, \varepsilon(s_1,\dots, s_{k+1}, s_m)$$
$$\times\ (e^{Ct} f_{r_n}, B_0 e^{Ct} g_{s_m})\, F^t_{\{r\}_{k+1}^{p'}, \overline{\{s\}_{k+1}^{q'}}}$$

(the primes mean that the r_n row and the s_m column are to be removed from the matrix).

To prove (A.4), we notice that, due to (A.2), we have to show that

$$Y_B[X] = Y_B\left[\prod_{i=1}^{p} a^\dagger(f_i)\right]\prod_{j=1}^{q} a(g_j) + \prod_{i=1}^{p} a^\dagger(f_i)\, Y_B\left[\prod_{j=1}^{q} a(g_j)\right]$$
$$+ \sum_{\{i\}_1^p}\sum_{\overline{\{j\}_1^q}} \varepsilon(\{i\}_1^p)\,\varepsilon(\overline{\{j\}_1^q})(f_{i_1}, B_0 g_{j_1}) \prod_{r=2}^{p} a^\dagger(f_{i_r}) \prod_{s=2}^{q} a(g_{j_s}) \tag{A.8}$$

To this purpose, we proceed by induction. We want to get that

$$Y_B\left[\prod_{i=1}^{p} a^\dagger(f_i)\prod_{j=1}^{q} a(g_j)\prod_{l=1}^{m} a^\dagger(f_{p+l})\right]$$
$$= Y_B\left[\prod_{i=1}^{p} a^\dagger(f_i)\right]\prod_{j=1}^{q} a(g_j)\prod_{l=1}^{m} a^\dagger(f_{p+l})$$
$$+ \prod_{i=1}^{p} a^\dagger(f_i)\, Y_B\left[\prod_{j=1}^{q} a(g_j)\right]\prod_{l=1}^{m} a^\dagger(f_{p+l})$$
$$+ \prod_{i=1}^{p} a^\dagger(f_i)\prod_{j=1}^{q} a(g_j)\, Y_B\left[\prod_{l=1}^{m} a^\dagger(f_{p+l})\right]$$
$$+ \sum_{\{i\}_1^p}\sum_{\overline{\{j\}_1^q}} \varepsilon(\{i\}_1^p)\,\varepsilon(\overline{\{j\}_1^q})(f_{i_1}, B_0 g_{j_1})$$
$$\times \prod_{r=2}^{p} a^\dagger(f_{i_r}) \prod_{s=2}^{q} a(g_{j_s}) \prod_{l=1}^{m} a^\dagger(f_{p+l}) \tag{A.9}$$

for any $m \in \mathbf{N}$.

This holds for $p=1$, because we can bring $a^\dagger(f_1)$ to the right of $\prod_{j=1}^{q} a(g_j)$, apply $Y_B[\cdot]$ according to (A.2), and then replace the first creation operator to the left of the product of annihilation operators.

Doing this, we obtain many terms because of the anticommutation relations.

Most of them cancel each other and we finally remain with the required expression.

Once we assume (A.9) to be valid for $p=n-1$, showing that it also holds for $p=n$ involves the same trick applied to the nth creation operator on the left of $\prod_{j=1}^{q} a(g_j)$.

Formula (A.9) implies (A.8), since we are free to choose m, i.e., the number of creation operators standing at the right of $\prod_{j=1}^{q} a(g_j)$.

APPENDIX B

Let us consider the matrix algebra M_d generated by creation and annihilation operators located at the points in an interval of size d around the origin.

Let us indicate with $(i)_k$ an increasing sequence of k points chosen among those of the given interval and use $\{a_i^\dagger, a_j\} = \delta_{ij}$ to write any element X belonging to M_d as follows:

$$X = C_0 + \sum_{k=1}^{d} \sum_{(i)_k} C_{(i)_k,0} a_i^\dagger \cdots a_{i_k}^\dagger + \sum_{k=1}^{d} \sum_{(i)_k} C_{0,(j)_k} a_{j_1} \cdots a_{j_k}$$
$$+ \sum_{p=1}^{d} \sum_{q=1}^{d} \sum_{(i)_p} \sum_{(j)_q} C_{(i)_p,(j)_q} a_{i_1}^\dagger \cdots a_{i_p}^\dagger a_{j_1} \cdots a_{j_q} \tag{B1}$$

Due to Appendix B and with ϕ the tracial state, we make the following observations that enable us to control

$$\|\gamma_t[a_{i_1}^\dagger \cdots a_{i_p}^\dagger a_{j_1} \cdots a_{j_q}] - \phi(a_{i_1}^\dagger \cdots a_{i_p}^\dagger a_{j_1} \cdots a_{j_q})\| \tag{B2}$$

Observations.

B2i.

$$a_r^\#(t) = a^\#(e^{Ct}\delta_r), \qquad \delta_r(\theta) = e^{-i\theta r}$$

$$\|a_r^\#(t)\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-4\gamma(\theta)t} \leqslant e^{-4\gamma t}$$

[either $\gamma \equiv \gamma(\theta)$ or $\gamma \equiv \min_{-\pi \leqslant \theta \leqslant \pi} \gamma(\theta)$].

B2ii. The symbols $\{r\}_k^p$, $\overline{\{s\}}_k^q$ appearing in the explicit expression of

$$\gamma_t[a_{i_1}^\dagger \cdots a_{i_p}^\dagger a_{j_1} \cdots a_{j_q}]$$

are the permutations of the numbers $i_1,..., i_p$ and $j_1,..., i_q$ respectively, which have been defined in Appendix A. We have $\binom{p}{k}$ and $\binom{q}{k}$ of such permutations, respectively.

B2iii. If $\gamma(\theta) \equiv \gamma > 0$, then

$$F_{ij}^t = \int_0^t ds(e^{Cs}\delta_i, B_0 e^{Cs}\delta_j) = \frac{\delta_{i,j}}{2}(1 - e^{-4\gamma t})$$

and

$$F_{\{r\}_k^p, \overline{\{s\}}_k^q}^t = \det \begin{bmatrix} F_{r_k s_1}^t & \cdots & F_{r_k s_k}^t \\ \vdots & & \vdots \\ F_{r_1 s_1}^t & \cdots & F_{r_1 s_k}^t \end{bmatrix}$$

is zero unless $r_i = s_i$, $i = 1,..., k$. In that case

$$|F_{\{r\}_k^p, \overline{\{s\}}_k^q}^t| = \frac{1}{2^k}(1 - e^{-4\gamma t})^k \leqslant \frac{1}{2^k}$$

B2iv. If $\gamma(\theta)$ is not constant but has a positive minimum γ, then

$$F^t_{rs} = \frac{2}{2}\left[\delta_{r,s} - \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\theta\, e^{i\theta(r-s)} e^{-4\gamma(\theta)t}\right]$$

and

$$|F^t_{\{r\}^p_k, \overline{\{s\}^q_k}}| \leqslant \frac{k!}{2^k}$$

Furthermore, when $\{r\}^p_p$ differs from $\overline{\{s\}^p_p}$ we get

$$|F^t_{\{r\}^p_p, \overline{\{s\}^p_p}}| \leqslant \frac{p!}{2^p} e^{-4\gamma t}$$

B2v. If $\{r\}^p_p = \overline{\{s\}^p_p}$, then

$$|F^t_{\{r\}^p_p, \{r\}^p_p} - \phi(a^\dagger_{r_1} \cdots a^\dagger_{r_p} a_{r_1} \cdots a_{r_p})|$$
$$\leqslant \begin{cases} \dfrac{p}{2^p} e^{-4\gamma t} & \text{when} \quad \gamma(\theta) \equiv \gamma \\ \dfrac{p!}{2^p} e^{-4\gamma t} & \text{when} \quad \gamma(\theta) \text{ has positive minimum} \end{cases}$$

Taking into account the above B2i–B2v, we can now discuss the various cases:

B3i. $p \neq q$:

$$\|\gamma_t[a^\dagger_{i_1} \cdots a^\dagger_{i_p} a_{j_1} \cdots a_{j_q}] - \phi(a^\dagger_{i_1} \cdots a^\dagger_{i_p} a_{j_1} \cdots a_{j_q})\|$$
$$\leqslant \sum_{k=0}^{[q,p]} \sum_{\{r\}^p_k} \sum_{\overline{\{s\}^q_k}} |F^t_{\{r\}^p_k, \overline{\{s\}^q_k}}|\, e^{-2\gamma(p+q-2k)t}$$
$$\leqslant e^{-2\gamma t} \sum_{k=0}^{[q,p]} \frac{k!}{2^k} \binom{p}{k} \binom{q}{k} \leqslant d! \left(\frac{3}{2}\right)^d e^{-2\gamma t}$$

B3ii. $p = q$, $(i)_p \neq (j)_p$:

$$\|\gamma_t[a^\dagger_{i_1} \cdots a^\dagger_{i_p} a_{j_1} \cdots a_{j_q}] - \phi(a^\dagger_{i_1} \cdots a^\dagger_{i_p} a_{j_1} \cdots a_{j_p})\|$$
$$\leqslant \sum_{k=0}^{p-1} \binom{p}{k}^2 \frac{k!}{2^k} e^{-4\gamma t(p-k)} + \frac{p!}{2^p} e^{-4\gamma t} \leqslant d! \left(\frac{3}{2}\right)\bigg)^d e^{-2\gamma t}$$

B3iii. $p=q,\ (i)_p=(j)_q$:

$$\|\gamma_t[a_{i_1}^\dagger\cdots a_{i_p}^\dagger a_{i_1}\cdots a_{i_p}]-\phi(a_{i_1}^\dagger\cdots a_{i_p}^\dagger a_{i_1}\cdots a_{i_p})\|$$

$$\leqslant\begin{cases}\displaystyle\sum_{k=0}^{p-1}\binom{p}{k}\frac{1}{2^k}e^{-4\gamma(p-k)t}+\frac{p}{2^p}e^{-4\gamma t}\leqslant e^{-4\gamma t}d\left(\frac{3}{2}\right)^d\cdots\gamma(\theta)\equiv\gamma\\ \displaystyle\sum_{k=0}^{p-1}\frac{k!}{2^k}\binom{p}{k}^2 e^{-4\gamma(p-k)t}+\frac{p!}{2^p}e^{-4\gamma t}\leqslant d!\left(\frac{3}{2}\right)^d e^{-4\gamma t}\cdots\min_{\pi\leqslant\theta\leqslant\pi}\gamma(\theta)=\gamma\end{cases}$$

Due to the fact that we have 4^d coefficients $C_{(i)_p,(j)_q}$, $C_{(i)_p,0}$, and $C_{0,(j)_q}$, each bounded in modulus by $\|X\|$, we conclude that

$$\begin{aligned}\|\gamma_t[X]-\phi[X]\|\leqslant&\left\{\sum_{k=1}^{d}\sum_{(i)_k}|C_{(i)_k,0}|+\sum_{k=1}^{d}\sum_{(j)_k}|C_{0,(j)_k}|\right.\\ &\left.+\sum_{p=1}^{d}\sum_{q=1}^{d}\sum_{(i)_p}\sum_{(j)_q}|C_{(i)_p,(j)_q}|\right\}\left(\frac{3}{2}\right)^d d!\,e^{-2\gamma t}\\ \leqslant&\,6^d d!\,e^{-2\gamma t}\|X\|\end{aligned}\tag{B3}$$

APPENDIX C

Fix the subalgebra N to be M_d, i.e., the subalgebra supported by a d-points interval around the origin. It is a $2^d\times 2^d$ matrix algebra and the tracial state ϕ restricted to it is represented by the density matrix $\tau=\mathbf{1}\cdot 2^{-d}$.

Let ρ_1 denote the density matrix $\hat{\phi}_i\circ\gamma^k|_N$, and ρ_2 the density matrix $\hat{\phi}_i\circ\gamma^{k-1}|_N$.

Use the integral representation

$$x\ln x=\int_0^{+\infty}d\alpha\left(x\frac{1}{1+\alpha}-x\frac{1}{x+\alpha}\right)$$

to get

$$\begin{aligned}&S(\phi|_N|\hat{\phi}_i\circ\gamma^k|_N)-S(\phi|_N|\hat{\phi}_i\circ\gamma^{k-1}|_N)\\ &=S(\tau|\rho_1)-S(\tau|\rho_2)\\ &=\mathrm{Tr}\int_0^{+\infty}d\alpha\left(\rho_2\frac{1}{\alpha+\rho_2}-\rho_1\frac{1}{\alpha+\rho_1}\right)\end{aligned}\tag{C1}$$

Since we know that ρ_1 and ρ_2 tend uniformly to τ with k, we can fix $K \in \mathbf{N}$ to be $\gg 1$ and for $k \geqslant K$ consider the expansion

$$\begin{aligned}
\rho_i \frac{1}{\alpha+\rho_i} &= \rho_i \frac{1}{\alpha+(\rho_i-\tau)+\tau} \\
&= \rho_i \frac{1}{\alpha+\tau}\left[1-\frac{1}{\alpha+\tau}(\rho_i-\tau)+\frac{1}{(\alpha+\tau)^2}(\rho_i-\tau)^2-\cdots\right] \\
&= \frac{\tau}{\alpha+\tau}-\frac{\tau}{\alpha+\tau}(\rho_i-\tau)+\frac{\tau}{(\alpha+\tau)^3}(\rho_i-\tau)^2 \\
&\quad +\frac{1}{\alpha+\tau}(\rho_i-\tau)-\frac{1}{(\alpha+\tau)^2}(\rho_i-\tau)^2+\cdots
\end{aligned} \tag{C2}$$

Since $\rho_i-\tau$ is Hermitian,

$$\operatorname{Tr}[\rho_i-\tau]^n = \|\rho_i-\tau\|_n^n$$

where $\|\cdot\|_n$ indicates the nth norm. Note that $\|\cdot\|_2$ is the Hilbert–Schmidt norm and $\|\cdot\|_n \leqslant \|\cdot\|_m \leqslant \cdots \leqslant \|\cdot\|_2 \leqslant \|\cdot\|_1$ for $n \geqslant m$. This and the uniform convergence argument quoted above allow us to state

$$\begin{aligned}
&S(\tau \mid \rho_1) - S(\tau \mid \rho_2) \\
&\quad = 2^{d-1}(\|\rho_1-\tau\|_2^2 - \|\rho_2-\tau\|_2^2) + O(\|\rho_1-\tau\|_2^2, \|\rho_2-2\|_2^2)
\end{aligned} \tag{C3}$$

ACKNOWLEDGMENTS

We thank Prof. Araki, who pointed out that a diffusion process calls for different scaling and so initiated this research. We thank SISSA, Trieste, and the Institute for Theoretical Physics of the University of Vienna for their hospitality, and especially Prof. Strocchi and Prof. Thirring for their continuing interest.

REFERENCES

1. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer, Berlin, 1982).
2. A. Connes and E. Størmer, *Acta Math.* **134**:289 (1975).
3. A. Connes, *C. R. Acad. Sci. Paris* **301**:1 (1985).
4. A. Connes, H. Narnhofer, and W. Thirring, *Commun. Math. Phys.* **112**:691 (1987).
5. H. Narnhofer and W. Thirring, *Lett. Math. Phys.* **14**:89 (1987).

6. T. Hudetz, Space time dynamical entropy of quantum systems, University of Vienna preprint UWThPh-1988-6; *Lett. Math. Phys.*, to appear.
7. S. Goldstein, *Commun. Math. Phys.* **39**:303 (1975).
8. H. Narnhofer and W. Thirring, *Lett. Math. Phys.* **15**:261 (1988).
9. E. Lieb and D. W. Robinson, *Commun. Math. Phys.* **28**:251 (1972).
10. E. B. Davies, *Commun. Math. Phys.* **55**:231 (1977).
11. E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1976).
12. G. Lindblad, *Commun. Math. Phys.* **48**:119 (1976).
13. F. Benatti, Thesis, SISSA, Trieste, Italy (1988).